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Weisner's method to obtain generating relations for the generalized polynomial set

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Abstract. In this paper, Weisner's group theoretic method is utilized to obtain the generating relations for the generalized hypergeometric polynomial set $\mathcal{U}_n(\beta; \gamma; x)$. To derive the elements of Lie algebra, a suitable interpretation to the index n is given. The generating relations are followed by its applications to the classical orthogonal polynomials, namely the Laguerre, Meixner, Gottlieb, Krawtchouk and Meixner–Pollaczek polynomials. Many results obtained as applications are known but some of those presented here are believed to be new.

1. Introduction

Recently, Bajpai and Arora [1] studied some properties of the generalized polynomial set $\mathcal{U}_n(\beta; \gamma; x)$, such as semi-orthogonality and an integral involving Fox's H-function. Hypergeometric polynomials (and hypergeometric series) in one and more variables arise naturally and rather frequently in a wide variety of problems in theoretical physics, applied mathematics, engineering sciences, statistics and operations research. In fact, a considerable field of physical and quantum mechanical situations (such as Schrödinger's wave mechanics) and various types of distributions in probability theory lead naturally to such classical orthogonal polynomials as the Laguerre, Meixner, Gottlieb, Krawtchowk and Meixner–Pollaczek polynomials. Also, the theory of generating relations for generalized hypergeometric polynomials play an important role in the problems of mathematical physics.

The principle objective of this paper is to derive some more interesting bilateral (or bilinear) generating relations for $\mathcal{U}_n(\beta; \gamma; x)$ using Weisner's group-theoretic method [2] by giving an interpretation to the index n. The useful ness of this method is that it yields a set of, at least, three generating relations. In an entire investigation, for the generalized polynomial set $\mathcal{U}_n(\beta; \gamma; x)$, six generating relations are derived followed by its applications to the sets of classical orthogonal polynomials, namely the Laguerre, Meixner, Gottlieb, Krawtchouk and Meixner–Pollaczek polynomials [3]. Many results obtained as applications are known but some of those presented here are believed to be new.

2. Definition

Bajpai and Arora [1] studied some properties of the generalized hypergeometric polynomial set

$$\mathcal{U}_n(\beta; \gamma; x) = x^n {}_2F_1\left[-n, \beta; \gamma; \frac{1}{x}\right] \tag{2.1}$$

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where n is a non-negative integer and x is any non-zero complex variable. β, γ are independent of n for if β, γ are dependent upon n then many properties which are valid for β, γ independent of n fail to be valid. The aim of the present paper is to derive some more bilinear and bilateral generating relations by Weisner's group-theoretic method.

These polynomials $\mathcal{U}_n(\beta; \gamma; x)$ satisfy the following descending and ascending recurrence relations, respectively:

$$\mathrm{D}\mathcal{U}_n(x) = n\mathcal{U}_{n-1}(x) \tag{2.2}$$

$$\mathrm{D}\mathcal{U}_n(x) = \frac{1}{x(1-x)}\{(\gamma+n)\mathcal{U}_{n+1}(x) + [(n+\beta) - (\gamma+2n)x]\mathcal{U}_n(x)\}. \tag{2.3}$$

These two independent differential recurrence relations determine the linear ordinary differential equation

$$x(1-x)\mathrm{D}^2\mathcal{U}_n(x) - [(n+\beta-1) - (\gamma+2n-2)x]\mathrm{D}\mathcal{U}_n(x) - n(\gamma+n-1)\mathcal{U}_n(x) = 0 \tag{2.4}$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}x$. The proofs of these results are obvious.

Applications

(1)

$$\lim_{\beta\to\infty}\left\{\beta^{-n}\mathcal{U}_n\left(\beta; 1+\alpha; \frac{\beta}{x}\right)\right\} = \frac{n!}{(1+\alpha)_n}x^{-n}L_n^{(\alpha)}(x) \tag{2.5}$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial [4, p 200].

(2)

$$\mathcal{U}_n(-y; \gamma; (1-\rho^{-1})^{-1}) = (1-\rho^{-1})^{-n}M_n(y; \gamma, \rho) \tag{2.6}$$

provided

$$\gamma > 0 \qquad 0 < \rho < 1 \qquad y = 0, 1, 2, \ldots$$

where $M_n(y; \gamma, \rho)$ is the Meixner polynomial [5, p 75].

(3)

$$\mathcal{U}_n(-y; 1; (1-\mathrm{e}^{\lambda})^{-1}) = (\mathrm{e}^{-\lambda}-1)^{-n}\phi_n(y, \lambda) \tag{2.7}$$

where $\phi_n(y, \lambda)$ is the Gottlieb polynomial [4, p 303].

(4)

$$\mathcal{U}_n(-y; -N; P) = P^n K_n(y; P, N) \tag{2.8}$$

provided

$$0 < P < 1 \qquad y = 0, 1, 2, 3, \ldots, N$$

where $K_n(y; P, N)$ is the known Krawtchouk polynomial [5, p 75].

(5)

$$\mathcal{U}_n(\lambda + \mathrm{i}y; 2\lambda; (1-\mathrm{e}^{-2\mathrm{i}\phi})^{-1}) = \frac{n!(2\mathrm{i})^{-n}}{(2\lambda)_n}\mathrm{cosec}^n\,\phi P_n^{\lambda}(y; \phi) \tag{2.9}$$

where $p_n^{\lambda}(y; \phi)$ is the known Meixner–Pollaczek polynomial [6, p 221].

(6)

$$\lim_{\beta\to\infty}\left\{\beta^{-n}\mathcal{U}_n\left(\beta; 1+\alpha-n; \frac{\beta}{x}\right)\right\} = \frac{(-1)^n}{(1+\alpha-n)_n}C_n(\alpha; x) \tag{2.10}$$

where $C_n(\alpha; x)$ is the known Poisson–Charlier polynomial [5, p 41].

Remark. One must be careful in deriving the generating relations for the Poisson–Charlier polynomials with the help of the above result. Here α is dependent upon n which should be considered when obtaining its applications.

3. Group-theoretic discussion

Let us write the differential equation (2.4) in operator functional notation as

$$L\left(x, \frac{\mathrm{d}}{\mathrm{d}x}, n\right) = x(1-x)\mathrm{D}^2 - [(n+\beta-1) - (\gamma+2n-2)x]\mathrm{D} - n(\gamma+n-1). \tag{3.1}$$

In order to use Weisner's method, we now construct from (2.4) the following partial differential equation by replacing $\mathrm{d}/\mathrm{d}x$ by $\partial/\partial x$, n by $y\partial/\partial y$ and $\mathcal{U}_n(x)$ by $u(x, y)$:

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + (2x-1)y\frac{\partial^2}{\partial x \partial y} - y^2\frac{\partial^2}{\partial y^2} + \left\{(\gamma-2)x - \beta + 1\right\}\frac{\partial}{\partial x} - \gamma y\frac{\partial}{\partial y}\right]u(x, y) = 0 \tag{3.2}$$

where $u(x, y) = y^n\mathcal{U}_n(x)$ is a solution of (3.2).

Let L represent the partial differential operator of (3.2), i.e.

$$L = L\left(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right) \equiv x(1-x)\frac{\partial^2}{\partial x^2} + (2x-1)y\frac{\partial^2}{\partial x\partial y} - y^2\frac{\partial^2}{\partial y^2} + \left\{(\gamma-2)x - \beta + 1\right\}\frac{\partial}{\partial x} - \gamma y\frac{\partial}{\partial y}. \tag{3.3}$$

We now seek linearly-independent lowering and raising differential operators B and C each of the form

$$A_1(x, y)\frac{\partial}{\partial x} + A_2(x, y)\frac{\partial}{\partial y} + A_3(x, y)$$

such that

$$B[y^n\mathcal{U}_n(\beta; \gamma; x)] = a_n\mathcal{U}_{n-1}(\beta; \gamma; x)y^{n-1}$$
$$C[y^n\mathcal{U}_n(\beta; \gamma; x)] = b_n\mathcal{U}_{n+1}(\beta; \gamma; x)y^{n+1} \tag{3.4}$$

where a_n and b_n are expressions in n which are independent of x and y, but not necessarily of the parameters β and γ. Each $A_i(x, y)$, $i = 1, 2, 3$, on the other hand, is an expression in x and y which is independent of n, but not necessarily of the parameters β and γ.

This necessitates the bringing into use of the recurrence relations (2.2) and (2.3). With the help of (2.2) and (2.3), it follows from (3.4) that

$$C = xy(1-x)\frac{\partial}{\partial x} + (2x-1)y^2\frac{\partial}{\partial y} + (\gamma x - \beta)y$$
$$B = -y^{-1}\frac{\partial}{\partial x}. \tag{3.5}$$

To find the group of operators, let us write $A \equiv y\partial/\partial y$.

Then we have the operators A, B and C which satisfy the following commutator relations:

$$[A, B] = -B \qquad [A, C] = C \qquad [B, C] = -2A - \gamma. \tag{3.6}$$

Now, every linear differential operator of the first order generates a one-parameter Lie group [2, p 27]; therefore, these commutator relations show that the set of operators $\{1, A, B, C\}$ generate a three-parameter Lie group [7].

Furthermore, we would like to prove that these operators commute with the partial differential operator L. We express L in terms of these operators.

We know that

$$Lu = x(1-x)\frac{\partial^2 u}{\partial x^2} + (2x-1)y\frac{\partial^2 u}{\partial x \partial y} - y^2\frac{\partial^2 u}{\partial y^2} + [(\gamma-2)x - \beta + 1]\frac{\partial u}{\partial x} - \gamma y\frac{\partial u}{\partial y}$$

and

$$CBu = -x(1-x)\frac{\partial^2 u}{\partial x^2} - (2x-1)y\frac{\partial^2 u}{\partial x \partial y} - [(\gamma-2)x - \beta + 1]\frac{\partial u}{\partial x}.$$

We get

$$[L + CB]u = -A^2 u - (\gamma - 1)Au.$$

Therefore,

$$Lu = -[CB + A^2 + (\gamma - 1)A]u. \tag{3.7}$$

By using the commutator relations we prove that the operators A, B and C commute with L and hence with $R = r_1 A + r_2 B + r_3 C + r_4$ where each r_i $(i = 1, 2, 3, 4)$ is an arbitrary constant, R is the set of differential operators.

This Lie algebra determines a root system and a Weyl group. The extended form of the group generated by each of the operators A, B and C is as follows:

$$\mathrm{e}^{aA} f(x, y) = f(x, \mathrm{e}^a y) \tag{3.8}$$

$$\mathrm{e}^{bB} f(x, y) = f\left(x - \frac{b}{y}, y\right) \tag{3.9}$$

$$\begin{aligned}\mathrm{e}^{cC} f(x, y) = {} & (1 - cxy)^{\beta-\gamma}[1 + cy(1-x)]^{-\beta} \\ & \times f\left(x[1 + cy(1-x)], \frac{y}{(1 - cxy)[1 + cy(1-x)]}\right)\end{aligned} \tag{3.10}$$

where a, b and c are arbitrary constants and $f(x, y)$ is an arbitrary function.

Then it is evident that

$$\mathrm{e}^{cC}\, \mathrm{e}^{bB} f(x, y) = (1 - cxy)^{\beta-\gamma}[1 + cy(1-x)]^{-\beta} f(\xi, n) \tag{3.11}$$

where

$$\xi = \{1 + cy(1-x)\}\left\{\frac{xy - b(1 - cxy)}{y}\right\}$$

$$\eta = \frac{y}{\{1 + cy(1-x)\}\{1 - cxy\}}.$$

4. Generating functions

4.1. Derivation from the operator $(A - \nu)$

We see that A generates a trivial group. Since $u(x, y) = y^\nu \mathcal{U}_\nu(x)$ is a solution of the simultaneous equations $Lu = 0$ and $(A - \nu)u = 0$ for arbitrary ν. Therefore we determine generating functions of $\mathcal{U}_n(x)$ by finding $\mathrm{e}^{bB+cC}[y^\nu \mathcal{U}_\nu(x)]$.

We need to consider three cases.

Case 1. Suppose $b = 1$, $c = 0$. Since for an arbitrary function $f(x, y)$

$$e^B f(x, y) = f\left(x - \frac{1}{y}, y\right)$$

we find

$$e^B f(x, y) = y^\nu \mathcal{U}_\nu(x - y^{-1}). \tag{4.1}$$

Since $B[y^\nu \mathcal{U}_\nu(x)] = (-\nu) y^{\nu-1} \mathcal{U}_{\nu-1}(x)$, we have

$$e^B [y^\nu \mathcal{U}_\nu(x)] = \sum_{n=0}^{\nu} \frac{(-\nu)_n}{n!} \mathcal{U}_{\nu-n}(x) y^{\nu-n}. \tag{4.2}$$

Equating expressions (4.1) and (4.2) and replacing y^{-1} by t, we get

$$\mathcal{U}_\nu(x - t) = \sum_{n=0}^{\nu} \frac{(-\nu)_n}{n!} \mathcal{U}_{\nu-n}(x) t^n. \tag{4.3}$$

Case 2. Suppose $b = 0$, $c = 1$. In this case we have

$$e^C [y^\nu \mathcal{U}_\nu(x)] = (1 - xy)^{\beta-\gamma-\nu} \{1 + y(1 - x)\}^{-\beta-\nu} y^\nu \mathcal{U}_\nu\{x + xy(1 - x)\}. \tag{4.4}$$

On the other hand,

$$e^C [y^\nu \mathcal{U}_\nu(x)] = \sum_{n=0}^{\infty} \frac{(\gamma + \nu)_n}{n!} \mathcal{U}_{\nu+n}(x) y^{\nu+n}. \tag{4.5}$$

Equating expressions (4.4) and (4.5) and replacing y by t, we get

$$(1 - xt)^{\beta-\gamma-\nu} \{1 + t(1 - x)\}^{-\beta-\nu} \mathcal{U}_\nu\{x + xt(1 - x)\} = \sum_{n=0}^{\infty} \frac{(\gamma + \nu)_n}{n!} \mathcal{U}_{\nu+n}(x) t^n. \tag{4.6}$$

Case 3. Suppose $bc \neq 0$, without any loss of generality we can choose $c = 1$ and $b = -1/w$ so we have

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\nu} & \frac{(-\nu)_k (\gamma + \nu - k)_n}{n! k!} \left(-\frac{1}{w}\right)^k \mathcal{U}_{\nu+n-k}(x) y^{n-k} \\ &= (1 - xy)^{\beta-\gamma-\nu} \{1 + y(1 - x)\}^{-\beta-\nu} \mathcal{U}_\nu \\ &\quad \times \left\{ \frac{[1 + y(1 - x)][xy + w^{-1}(1 - xy)]}{y} \right\}. \end{aligned} \tag{4.7}$$

Applications. Furthermore, using the conditions of section 2 in (4.3) and (4.6) we have:

(1)

$$\sum_{n=0}^{\nu} \frac{(-\alpha - \nu)_n L_{\nu-n}^{(\alpha)}(x) t^n}{n!} = (1 - t)^\nu L_\nu^{(\alpha)}\left(\frac{x}{1 - t}\right).$$

(2)

$$\sum_{n=0}^{\infty} \frac{(\nu + n)! L_{\nu+n}^{(\alpha)}(x) t^n}{\nu! n!} = (1 - t)^{-1-\alpha-\nu} \exp\left[\frac{-xt}{1 - t}\right] L_\nu^{(\alpha)}\left(\frac{x}{1 - t}\right).$$

Here if $\nu = 0$, we have

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1 - t)^{-1-\alpha} \exp\left[\frac{-xt}{1 - t}\right].$$

(3)

$$\sum_{n=0}^{\nu} \frac{(-\nu)_n (1-\rho^{-1})^n M_{\nu-n}(y;\gamma,\rho) t^n}{n!}$$
$$= \{1 - t(1-\rho^{-1})\}^{\nu} M_\nu \left(y; \gamma, \left(\frac{1 - t(1-\rho^{-1})}{\rho^{-1} - t(1-\rho^{-1})} \right) \right)$$

provided

$$\gamma > 0 \qquad 0 < \rho < 1 \qquad y = 0, 1, 2, \ldots .$$

(4)

$$\sum_{n=0}^{\nu} \frac{(\gamma+\nu)_n (1-\rho^{-1})^{-n} M_{\nu+n}(y;\gamma,\rho) t^n}{n!}$$
$$= \{1 - t(1-\rho^{-1})^{-1}\}^{-(y+\nu+\gamma)} \{1 - t\rho^{-1}(1-\rho^{-1})^{-1}\}^{y}$$
$$\times M_\nu \left(y; \gamma, \left(\frac{\rho - 1 - t}{1 - t - \rho^{-1}} \right) \right)$$

provided

$$\gamma > 0 \qquad 0 < \rho < 1 \qquad y = 0, 1, 2, \ldots .$$

(5)

$$\sum_{n=0}^{\nu} \frac{(-\nu)_n (\mathrm{e}^{-\lambda} - 1)^n \phi_{\nu-n}(y;\lambda) t^n}{n!}$$
$$= \{1 - t(\mathrm{e}^{-\lambda} - 1)\}^{\nu} \phi_\nu \left(y, \log \left(\frac{\mathrm{e}^{\lambda}(1-\mathrm{e}^{\lambda})^{-1} - t}{(1-\mathrm{e}^{\lambda})^{-1} - t} \right) \right).$$

(6)

$$\sum_{n=0}^{\infty} \frac{(\nu+1)_n (\mathrm{e}^{-\lambda} - 1)^{-n} \phi_{\nu+n}(y,\lambda) t^n}{n!}$$
$$= \{1 - t(1-\mathrm{e}^{-\lambda})^{-1}\}^{-y-1} \{1 - t\,\mathrm{e}^{\lambda}(1-\mathrm{e}^{\lambda})^{-1}\}^{y-\nu}$$
$$\cdot \phi_\nu \left(y, \log \left(\frac{1 - t - \mathrm{e}^{\lambda}}{(1-\mathrm{e}^{\lambda})^{-1} - t} \right) \right).$$

(7)

$$\sum_{n=0}^{\infty} \frac{(-\nu)_n P^{-n} K_{\nu-n}(y;P,N) t^n}{n!} = \left(1 - \frac{t}{P}\right)^{\nu} K_\nu(y; P - t, N)$$

provided

$$0 < P < 1 \qquad y = 0 \qquad 1, 2, \ldots, N.$$

(8)

$$\sum_{n=0}^{\infty} \frac{(\nu - N)_n P^n K_{\nu+n}(y;P,N) t^n}{n!}$$
$$= (1 - Pt)^{N-y-\nu} \{1 + t(1-P)\}^{y} K_\nu(y; P + Pt(1-P))$$

provided

$$0 < P < 1 \qquad y = 0, 1, 2, \ldots, N.$$

(9)

$$\sum_{n=0}^{\infty} \frac{(1-2\lambda-\nu)_n (2\mathrm{i}\sin\phi)^n P^{\lambda}_{\nu-n}(y;\phi) t^n}{n!} = P^{\lambda}_{\nu}\left[y, \frac{1}{2\mathrm{i}}\log\left(\frac{t-(1-\mathrm{e}^{-2\mathrm{i}\phi})^{-1}}{t-1-(1-\mathrm{e}^{-2\mathrm{i}\phi})^{-1}}\right)\right].$$

(10)

$$\sum_{n=0}^{\infty} \frac{(1+\nu)_n \operatorname{cosec}^n \phi P^{\lambda}_{\nu+n}(y;\phi) t^n}{n!(2\mathrm{i})^n} = P^{\lambda}_{\nu}\left[y, \frac{1}{2\mathrm{i}}\log\left(\frac{\mathrm{e}^{-2\mathrm{i}\phi}-1-t}{1-\mathrm{e}^{-2\mathrm{i}\phi}-t}\right)\right].$$

4.2. Derivation from operators not conjugate to $(A-\nu)$

Let $S = \mathrm{e}^{cC}\mathrm{e}^{bB}$, where b and c are arbitrary constants.

Now according to McBride [2] we find that

$$\begin{aligned}
\mathrm{e}^{bB} C\, \mathrm{e}^{-bB} &= -2bA - b^2 B + C - b\gamma\\
\mathrm{e}^{bB} A\, \mathrm{e}^{-bB} &= A + bB\\
\mathrm{e}^{cC} A\, \mathrm{e}^{-cC} &= A - cC\\
\mathrm{e}^{cC} B\, \mathrm{e}^{-cC} &= 2cA + B - c^2 C + c\gamma.
\end{aligned}$$

Consider the set of linear differential operators $\{R/R = r_1 A + r_2 B + r_3 C + r_4$, for all combinations of zero and non-zero coefficients except for $r_1 = r_2 = r_3 = 0\}$.

We find that

$$\begin{aligned}
S(A-\nu)S^{-1} &= \mathrm{e}^{cC}\,\mathrm{e}^{bB}(A-\nu)\,\mathrm{e}^{-bB}\,\mathrm{e}^{-cC}\\
&= (1+2bc)A + bB - c(1+bc)C + (bc\gamma-\nu).
\end{aligned}$$

Then

$$r_1 = 1+2bc \qquad r_2 = b \qquad r_3 = -c(1+bc).$$

From two of these three equations we can find b and c in terms of r_1, r_2 and r_3. The third equation then imposes a restrictive relation on the r_i $(i = 1, 2, 3)$ which implies that $r_1^2 + 4r_2 r_3 = 1$.

Therefore, $(A-\nu)$ is not conjugate to operators for which $r_1^2 + 4r_2 r_3 = 0$.

We consider the following cases.

Case 1. If $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, we seek a solution of the system $Lu = 0$ and $(B+\lambda)u = 0$, where λ is a non-zero constant.

For convenience, we choose $\lambda = 1$ and write the equation as $Lu = 0$ and $(B+1)u = 0$.

A solution of this system is

$$u(x, y) = \mathrm{e}^{xy}\,{}_1F_1[\beta;\gamma;-y].$$

If this function is expanded in powers of y, we get

$$\mathrm{e}^{xy}\,{}_1F_1[\beta;\gamma;-y] = \sum_{n=0}^{\infty} \frac{\mathcal{U}_n(\beta;\gamma;x) y^n}{n!}. \tag{4.8}$$

Case 2. If $r_1 = 2$, $r_2 = 1$, $r_3 = -1$, we are led to this choice by considering $\mathrm{e}^{cC}(B-w)\,\mathrm{e}^{-cC}$, where w is a non-zero constant. We find that

$$\mathrm{e}^{cC}(B-w)\,\mathrm{e}^{-cC} = 2cA + B - c^2 C + (c\gamma - w).$$

If we let $c = 1$, we get $r_1 = 2$, $r_2 = 1$, $r_3 = -1$. Since this choice satisfies the above conditions, we may determine a solution of the system $Lu = 0$ and $(2A + B - C + \gamma - w)u = 0$. From the generating function of (4.8), by replacing y by $-wy$ we get

$$u(x, -wy) = \mathrm{e}^{-xyw}\,{}_1F_1[\beta;\gamma;wy].$$

We know that for an arbitrary function $f(x, y)$

$$e^C f(x, y) = (1-xy)^{\beta-\gamma}\{1+y(1-x)\}^{-\beta} f\left(x\{1+y(1-x)\}, \frac{y}{(1-xy)\{1+y(1-x)\}}\right).$$

Thus

$$\begin{aligned} e^C u(x, -wy) &= (1-xy)^{\beta-\gamma}\{1+y(1-x)\}^{-\beta} \\ &\quad \times \exp\left(\frac{-xyw}{1-xy}\right) {}_1F_1\left[\beta; \gamma; \frac{wy}{(1-xy)(1+y(1-x))}\right] \\ &= \sum_{n=0}^{\infty} \frac{(1-xy)^{\beta-\gamma-n}\{1+y(1-x)\}^{-\beta-n}(-wy^n)}{n!} \mathcal{U}_n\{x+xy(1-x)\} \\ &\quad \text{(by using 4.8)}. \end{aligned}$$

With the help of (4.6), we get

$$\begin{aligned} (1-xy)^{\beta-\gamma}\{1+y(1-x)\}^{-\beta} \exp\left(\frac{-xyw}{1-xy}\right) {}_1F_1\left[\beta; \gamma; \frac{wy}{(1-xy)\{1+y(1-x)\}}\right] \\ = \sum_{n=0}^{\infty} \mathcal{U}_n(\beta; \gamma; x) L_n^{(\gamma-1)}(w) y^n. \end{aligned} \tag{4.9}$$

Remark. The corresponding bilateral (or bilinear) generating relations for the Laguerre, Meixner, Gottlieb, Krawtchouk and Meixner–Pollaczek polynomials can be deduced from (4.9) by using the conditions of section 2.

Case 3. Let $r_1 = 0$, $r_2 = 0$, $r_3 = 1$. We seek the solution of the system $Lu = 0$ and $(C+\phi)u = 0$, where ϕ is a non-zero constant. We may avoid actually solving this system by noting that

$$e^{bB} e^{cC}(B+1) e^{-cC} e^{-bB} = 2c(1+bc)A + (1+bc)^2 B - c^2 C + \gamma c(1+bc) + 1.$$

If we choose $b = 1$, $c = -1$, we get

$$e^B e^{-C}(B+1) e^C e^{-B} = -C + 1.$$

Therefore, we can obtain a solution of $Lu = 0$ and $(C-1)u = 0$ by transforming the generating function (4.8) as

$$\begin{aligned} e^B e^{-C}\{e^{xy}{}_1F_1[\beta; \gamma, -y]\} \\ = y^{-\gamma} x^{\beta-\gamma}[-(1-x)]^{-\beta} \exp\left(1 - \frac{1}{xy}\right) {}_1F_1\left[\beta; \gamma; \frac{1}{xy(1-x)}\right]. \end{aligned}$$

If we let $y = -1/t$ and expand in powers of t we get

$$\exp\left(\frac{t}{x}\right) {}_1F_1\left[\beta; \gamma; \frac{-t}{x(1-x)}\right] = \sum_{n=0}^{\infty} \frac{\mathcal{U}_n(\beta; \gamma; 1-x)}{n!}\left[\frac{t}{x(1-x)}\right]^n. \tag{4.10}$$

Applications. The following results can be obtained from the generating function (4.8) by using the conditions of section 2.

(1)

$$\sum_{n=0}^{\infty} \frac{x^{-n} L_n^{(\alpha)}(x) y^n}{(1+\alpha)_n} = \exp(y/x)\,{}_0F_1[-; 1+\alpha; -y]$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x) z^n}{(1+\alpha)_n} = \exp(z)\,{}_0F_1[-; 1+\alpha; -xz].$$

(2)

$$\sum_{n=0}^{\infty} \frac{(1-\rho^{-1})^{-n} M_n(t;\gamma,\rho) y^n}{n!} = \exp\{y(1-\rho^{-1})^{-1}\}{}_1F_1[-t;\gamma;-y]$$

which is equivalent to

$$\sum_{n=0}^{\infty} \frac{M_n(t;\gamma,\rho) z^n}{n!} = \exp(z)\,{}_1F_1[-t;\gamma;-z(1-\rho^{-1})]$$

provided

$$\gamma > 0 \qquad 0 < \rho < 1 \qquad y = 0, 1, 2, \ldots .$$

(3)

$$\sum_{n=0}^{\infty} \frac{(\mathrm{e}^{-\lambda}-1)^{-n} \phi_n(t;\lambda) y^n}{n!} = \exp\{y(1-\mathrm{e}^{\lambda})^{-1}\}{}_1F_1[-t;1;-y]$$

which can be reduced to

$$\sum_{n=0}^{\infty} \frac{\phi_n(t;\lambda) z^n}{n!} = \exp(z)\,{}_1F_1[1+t;1;-z(1-\mathrm{e}^{-\lambda})].$$

(4)

$$\sum_{n=0}^{\infty} \frac{P^n K_n(t;P,N) y^n}{n!} = \exp(py)\,{}_1F_1[-t;-N;-y]$$

which is equivalent to

$$\sum_{n=0}^{\infty} \frac{K_n(t;P,N) z^n}{n!} = \exp(z)\,{}_1F_1[-t;-N;-zP^{-1}]$$

provided

$$0 < P < 1 \qquad y = 0, 1, 2, \ldots, N.$$

(5)

$$\sum_{n=0}^{\infty} \frac{(2\mathrm{i})^{-n} \operatorname{cosec}^n \phi\, P_n^{\lambda}(x;\phi) y^n}{n!} = \exp[y(1-\mathrm{e}^{2\mathrm{i}\phi})^{-1}]\,{}_1F_1[\lambda+\mathrm{i}x;2\lambda;-y]$$

which can be reduced to

$$\sum_{n=0}^{\infty} \frac{P_n^{\lambda}(x;\phi) z^n}{n!} = \exp[z\,\mathrm{e}^{\mathrm{i}\phi}]\,{}_1F_1[\lambda+\mathrm{i}x;2\lambda;-z\,\mathrm{e}^{\mathrm{i}\phi}(1-\mathrm{e}^{-2\mathrm{i}\phi})].$$

These are all well known generating functions in one form or another for the Laguerre, Meixner, Gottlieb, Krawtchouk and Meixner–Pollaczek polynomials, respectively.

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